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Plane Lorentzian and Fuchsian Hedgehogs

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Abstract. Parts of the Brunn–Minkowski theory can be extended to hedgehogs, which are envelopes of families of affine hyperplanes parametrized by their Gauss map. F. Fillastre introduced Fuchsian convex bodies, which are the closed convex sets of Lorentz–Minkowski space that are globally invariant under the action of a Fuchsian group. In this paper, we undertake a study of plane Lorentzian and Fuchsian hedgehogs. In particular, we prove the Fuchsian analogues of classical geometrical inequalities (analogues that are reversed as compared to classical ones).

1 Introduction

Our main results consist in the Fuchsian analogues of some classical geometrical inequalities. These results are presented in Subsection 1.3. For the convenience of the reader, we recall very briefly some definitions and results concerning plane Euclidean hedgehogs in Subsection 1.1. A short introduction to plane Lorentzian hedgehogs and first results concerning evolutes and duality in the Lorentz-Minkowski plane L^2 are presented in Subsection 1.2. Finally, Subsection 1.3 presents a study of plane Fuchsian hedgehogs (convolution of Fuchsian hedgehogs, Brunn-Minkowski and Minkowski type inequalities, reversed isoperimetric inequality, isometric excess and area of the evolute, reversed Bonnesen inequality).

1.1 Plane Euclidean Hedgehogs

In the Euclidean plane $\mathbb{R}^2$, a hedgehog is the envelope of a family of cooriented lines $L(\theta)$ parametrized by the oriented angle $\theta \in \mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ from $e_1 = (1,0)$ to their coorienting normal vector $u(\theta) = (\cos\theta, \sin\theta)$. These cooriented lines $L(\theta)$ have equations

$$\langle x, u(\theta)\rangle = h(\theta), \tag{1.1}$$

where $\langle \cdot, \cdot\rangle$ is the usual inner product on $\mathbb{R}^2$ and where $h \in C^1(\mathbb{S}^1;\mathbb{R})$. Partial differentiation of (1.1) yields

$$\langle x, u'(\theta)\rangle = h'(\theta). \tag{1.2}$$

From (1.1) and (1.2), the parametrization of the corresponding hedgehog is

$$x_h\colon \mathbb{S}^1 \to \mathbb{R}^2, \quad \theta \mapsto h(\theta)u(\theta) + h'(\theta)u'(\theta).$$

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This envelope $\mathcal{H}_h := x_h(\mathbb{S}^1)$ is called the (*Euclidean*) *hedgehog* with support function h. If h is only C^1, then $\mathcal{H}_h$ may be a fractal curve [9]. In this paper, we shall be mainly interested in C^2-*hedgehogs*, that is, hedgehogs with a C^2-support function. Note that regular C^2-hedgehogs of $\mathbb{R}^2$ are strictly convex smooth curves, and that any C^2-hedgehog can be regarded as the Minkowski difference of two such convex curves [10].

H. Geppert was the first to introduce hedgehogs in $\mathbb{R}^2$ and $\mathbb{R}^3$ (under the respective German names *stützbare Bereiche* and *stützbare Flächen)* in an attempt to extend parts of the Brunn–Minkowski theory [3]. Many classical inequalities for convex curves have counterparts for hedgehogs. Of course, a few adaptations are necessary. In particular, lengths and areas have to be replaced by algebraic versions. For instance, Theorem 1.1 extends the isoperimetric inequality and gives an upper bound of the isoperimetric deficit in terms of signed area of the evolute.

Theorem 1.1 ([8, Prop. 6]) *For any $h \in C^3(\mathbb{S}^1;\mathbb{R})$, we have*

$$0 \le l(h)^2 - 4\pi a(h) \le -4\pi a(h'), \tag{1.3}$$

where $l(h)$ and $a(h)$ are respectively the signed length and area of $\mathcal{H}_h$ and where $a(h')$ is the signed area of its evolute. In each inequality of (1.3), the equality holds if and only if $\mathcal{H}_h$ is a circle or a point.

In Section 6, we shall prove an analogue of Theorem 1.1 for Fuchsian hedgehogs. For a study of plane Euclidean hedgehogs, see [10]. An introduction to hedgehogs in higher dimensions is given in [6].

1.2 Plane Lorentzian Hedgehogs

In this paper, we shall undertake a similar study replacing the Euclidean plane $\mathbb{R}^2$ by the Lorentzian plane L^2 and the unit circle $\mathbb{S}^1$ of $\mathbb{R}^2$ by the hyperbolic line $\mathbb{H}^1$. In the Lorentzian plane L^2, a *spacelike hedgehog* is similarly defined to be the envelope of a family of cooriented spacelike lines $L(t)$ parametrized by the oriented hyperbolic angle $t \in \mathbb{H}^1 \simeq \mathbb{R}$ from $e_2 = (0,1)$ to their coorienting normal vector $v(t) = (\sinh t, \cosh t)$ (see Section 2). These cooriented lines $L(t)$ have equations

$$\langle x, v(t)\rangle_L := h(t), \tag{1.4}$$

where $\langle x, y\rangle_L := x_1y_1 - x_2y_2$ is the Lorentzian inner product of the vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in L^2, and where $h \in C^1(\mathbb{R};\mathbb{R})$. Note that $h(t)$ is the signed distance from the origin to the support line with coorienting unit normal $v(t)$. Partial differentiation of (1.4) yields

$$\langle x, v'(t)\rangle_L := h'(t). \tag{1.5}$$

From (1.4) and (1.5), the parametrization of the corresponding hedgehog is

$$x_h\colon \mathbb{H}^1 \to L^2, \quad t \mapsto h'(t)v'(t) - h(t)v(t).$$

This envelope $\mathcal{S}_h := x_h(\mathbb{H}^1)$ is called the *spacelike hedgehog* of L^2 with support function $h \in C^1(\mathbb{H}^1;\mathbb{R})$. We shall generally restrict discussion to C^2-hedgehogs (*i.e.,* with a C^2-support function). In Section 3, we shall give a study of their evolutes. In particular, we shall prove the following theorem.

Theorem 1.2 *For any $h \in C^3(\mathbb{R};\mathbb{R})$, the second evolute of $\mathcal{S}_h$ is the spacelike hedgehog with support function h'':*

$$\mathcal{D}(\mathcal{D}(\mathcal{S}_h)) = \mathcal{S}_{h''},$$

where $\mathcal{D}(\mathcal{E})$ denotes the evolute of the envelope $\mathcal{E} \subset L^2$ of a family of nonlightlike lines with a C^3-support function and no inflection point.

In Section 3, we shall also introduce timelike hedgehogs of L^2, and in Section 4, we shall give explicit formulas describing a natural duality relationship between spacelike hedgehogs and timelike hedgehogs.

For a systematic study of curves in the Lorentz–Minkowski plane, we refer the reader to [7, Subsection 2.3].

1.3 Plane Fuchsian Hedgehogs

Of course, a spacelike hedgehog $\mathcal{S}_h \subset L^2$ has no reason to be compact. So, in order to develop a Brunn–Minkowski theory, we shall replace $\mathbb{H}^1$ by its quotient by a Fuchsian group Γ. In other words:

(i) we identify

$$SO(1,1) = \left\{ M = \begin{pmatrix} x_2 & x_1 \\ x_1 & x_2 \end{pmatrix} \in M_2(\mathbb{R}) \mid x_2^2 - x_1^2 = 1 \right\}$$

with the hyperbola $H = \{(x_1, x_2) \in L^2 \mid x_2^2 - x_1^2 = 1\}$;

(ii) we take the subgroup Γ of $SO(1,1)$ that is generated by $(\sinh T, \cosh T) \in \mathbb{H}^1 = \{(x_1, x_2) \in H \mid x_2 > 0\}$ for some $T \in \mathbb{R}_+^*$;

(iii) we replace $\mathbb{H}^1$ by $\mathbb{H}^1/\Gamma \simeq \mathbb{R}/T\mathbb{Z}$.

In practice, any $h \in C^1(\mathbb{H}^1/\Gamma;\mathbb{R})$ will be regarded as a T-periodic function $h \in C^1(\mathbb{R};\mathbb{R})$. The Γ-*hedgehog* with support function $h \in C^1(\mathbb{H}^1/\Gamma;\mathbb{R})$ is then defined to be the curve Γ_h parametrized by

$$\gamma_h\colon \mathbb{R} \to L^2, \quad t \mapsto h'(t)v'(t) - h(t)v(t).$$

Note that, for any $t \in \mathbb{R}$, we have $\gamma_h(t+T) = g(T)[\gamma_h(t)]$, where $g(T)$ denotes the linear isometry of L^2 whose matrix in the canonical basis is

$$\begin{pmatrix} \cosh T & \sinh T \\ \sinh T & \cosh T \end{pmatrix}.$$

Minkowski differences of convex bodies of $\mathbb{R}^2$ do not only constitute a real vector space $(\mathcal{H}^2, +, \cdot)$ but also a commutative and associative $\mathbb{R}$-algebra. Indeed, H. Görtler [4, 5] defined the convolution product of two hedgehogs $\mathcal{H}_f$ and $\mathcal{H}_g$ of $\mathbb{R}^2$ as the hedgehog with support function

$$(f * g)(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta - \alpha) g(\alpha)\, d\alpha,$$

for all $\theta \in \mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$, and we can check at once that $(\mathcal{H}^2, +, \cdot, *)$ is then a commutative and associative algebra. The point of interest is that the convolution product of two Euclidean hedgehogs inherits many properties of the factors [10, Section 6]. In particular, H. Görtler noticed that the convolution product of two convex bodies of $\mathbb{R}^2$ is still a convex body. The purpose of Section 5 will be to give a Fuchsian analogue of Görtler's theorem. Note that Görtler's algebra should not be confused with McMullen's polytope algebra [11].

For every $h \in C^2(\mathbb{H}^1/\Gamma; \mathbb{R})$, the C^1-curve

$$\gamma_h : [0, T] \to L^2/\Gamma, \quad t \mapsto h'(t)\nu'(t) - h(t)\nu(t)$$

is rectifiable and its length is given by

$$L(h) := \int_0^T \|x_h'(t)\|_L dt,$$

where $\|x\|_L := \sqrt{|\langle x, x\rangle|_L}$ for all $x \in L^2$. Note that $x_h' = R_h \nu'$, where $R_h := h'' - h$ is the so-called *curvature function* of Γ_h. Therefore,

$$L(h) := \int_0^T |R_h(t)| dt.$$

If in this last integral we remove the absolute value to take into account the sign of the curvature function of Γ_h, we obtain the so-called *algebraic (or signed) length of* Γ_h, which is thus given by

$$l(h) := \int_0^T R_h(t) dt = -\int_0^T h(t) dt.$$

Given any $h \in C^2(\mathbb{H}^1/\Gamma; \mathbb{R})$, let Δ_h be the oriented closed curve in L^2 consisting of the oriented line segment joining the origin to $\gamma_h(0)$, followed by the oriented curve Γ_h and finally the oriented line segment joining $\gamma_h(T)$ to the origin. Denote by $(\Delta_h)^-$ the curve obtained from Δ_h by taking the opposite orientation (see Figure 1). Define the *algebraic (or signed) area* of Γ_h to be the algebraic area bounded by $(\Delta_h)^-$, that is,

$$a(h) := \int_{L^2} i_h(x) d\lambda(x),$$

where λ is the Lebesgue measure and $i_h(x)$ the winding number of x with respect to $(\Delta_h)^-$ for $x \in L^2 - (\Delta_h)^-$ (we let $i_h(x) = 0$ for $x \in (\Delta_h)^-$). An easy straightforward calculation gives

$$a(h) = \frac{1}{2}\int_{(\Delta_h)^-} x_1 dx_2 - x_2 dx_1 = \frac{1}{2}\int_0^T h(t) R_h(t) dt = \frac{1}{2}\int_0^T (h^2 + (h')^2)(t) dt.$$

In the Fuchsian case, many geometric inequalities will be reversed. A first example is given by the following obvious result.

Proposition 1.3 *The map* $\sqrt{a} : C^2(\mathbb{H}^1/\Gamma; \mathbb{R}) \to \mathbb{R}_+, h \longmapsto \sqrt{a(h)}$ *is a norm associated with a scalar product* $(h, k) \longmapsto a(h, k)$*. In particular, for any* $(h, k) \in C^2(\mathbb{H}^1/\Gamma; \mathbb{R})^2$*, we have*

$$\sqrt{a(h+k)} \le \sqrt{a(h)} + \sqrt{a(k)} \tag{1.6}$$

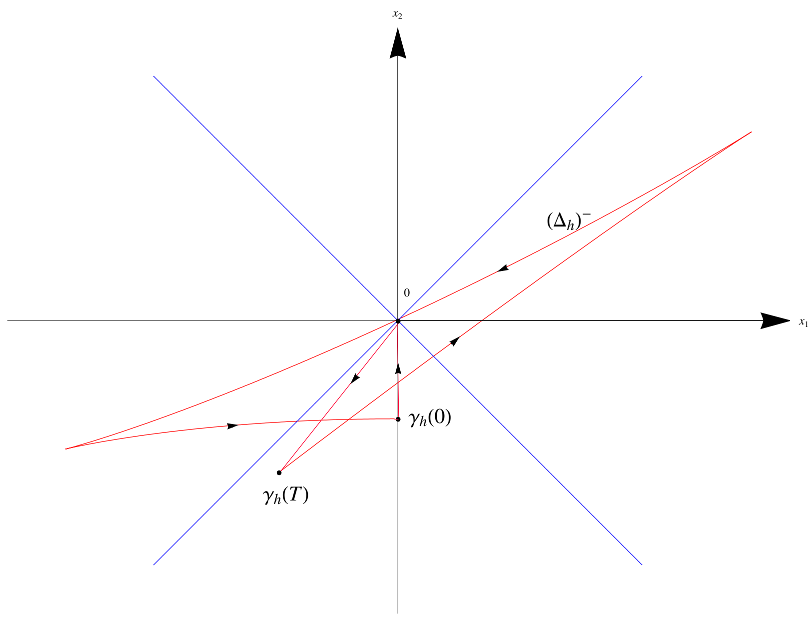

Figure 1: The curve $(\Delta_h)^-$ when $h(t) := 1 + \cos(2\pi t)$.

and

$$a(h,k)^2 \leq a(h)\,a(k), \tag{1.7}$$

with equalities if and only if Γ_h *and* Γ_k *are homothetic (here, "homothetic" means that there exists* $(\lambda,\mu) \in \mathbb{R}^2 - \{(0,0)\}$ *such that* $\lambda h + \mu k = 0$).

Indeed, inequality (1.6) (resp. (1.7)) has to be compared with the Brunn–Minkowski inequality (resp. Minkowski inequality) in $\mathbb{R}^2$ (*e.g.*, see [13, Section 7]). For any pair (H,K) of convex bodies of $\mathbb{R}^2$, we have

$$\sqrt{a(H+K)} \geq \sqrt{a(H)} + \sqrt{a(K)} \quad \text{and} \quad a(H,K)^2 \geq a(H)\,a(K),$$

where $a(L)$ (resp. $a(H,K)$) is the area (resp. the mixed area) of L (resp. (H,K)). By taking $k = -1$ (that is, $\Gamma_k = \mathbb{H}^1$) in (1.7), we obtain the following *reversed isoperimetric inequality*

$$a(h) \geq \frac{l(h)^2}{2T}, \tag{1.8}$$

with equality if and only if Γ_h and $\mathbb{H}^1$ are homothetic (that is, h is constant). In Section 6, another reversed geometric inequality will be given by the following analogous of Theorem 1.1 for Fuchsian hedgehogs.

Theorem 1.4 *Let $T \in]0, 2\pi]$. For any T-periodic $h \in C^3(\mathbb{R};\mathbb{R})$, we have*

$$0 \le 2Ta(h) - l(h)^2 \le 2Ta(h'),$$

where $l(h)$ and $a(h)$ are respectively the signed length and area of Γ_h and $a(h')$ is the signed area of its evolute.

Note that $2Ta(h) - l(h)^2$ provides a measure of how far Γ_h deviates from a Γ-hedgehog given by a spacelike branch of a hyperbola. A lower bound of the isoperimetric excess $a(h)-l(h)^2/2T$ is given by the following reversed Bonnesen inequality, which we shall prove in Section 7.

Theorem 1.5 (Reversed Bonnesen inequality) *For any T-periodic $h \in C^2(\mathbb{R};\mathbb{R})$,*

$$\frac{1}{2T}(R-r)^2 \le a(h) - \frac{l(h)^2}{2T},$$

where $l(h)$ and $a(h)$ are respectively the signed length and area of Γ_h and where $r := \min_{0\le t\le T}(-h(t))$ and $R := \max_{0\le t\le T}(-h(t))$. Furthermore, the equality holds if and only if $R = r$.

Recall that Bonnesen's sharpening of the isoperimetric inequality for a convex body K with non-empty interior in $\mathbb{R}^2$ reads as follows:

$$L^2 - 4\pi A \ge \pi^2(R-r)^2,$$

where L and A are respectively the perimeter and the area of K and where r and R stand respectively for the inradius and the circumradius of K (*e.g.,* see [1, pp. 108–110]).

For geometric inequalities involving hedgehogs in higher dimensions, see [8] for the Euclidean case and [2] for the Fuchsian case.

2 Preliminaries

In this paper, the notation $x = (x_1, x_2)$ means that (x_1, x_2) are the coordinates of $x \in \mathbb{R}^2$ with respect to the canonical basis of $\mathbb{R}^2$. The *Lorentzian plane* L^2 is the vector space $\mathbb{R}^2$ endowed with the pseudo-scalar product $\langle x, y\rangle_L := x_1y_1 - x_2y_2$, for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$. For any $x \in L^2$, define the *norm* of x by $\|x\|_L = \sqrt{|\langle x, x\rangle_L|}$ and the *sign of* x by $\varepsilon(x) = \operatorname{sgn}(\langle x, x\rangle_L)$, where sgn denotes the sign function: $\operatorname{sgn}(t)$ is 1, 0, or -1 if t is positive, zero, or negative, respectively. A nonzero vector $x \in L^2$ is said to be *spacelike* if $\varepsilon(x) = 1$, *lightlike* if $\varepsilon(x) = 0$, and *timelike* if $\varepsilon(x) = -1$. Let $e_2 = (0,1)$. A timelike vector $x = (x_1, x_2) \in L^2$ is said to be a *future vector* if $\langle x, e_2\rangle_L < 0$, that is, if $x_2 > 0$. We shall denote by F the set of all future timelike vectors:

$$F = \{x = (x_1, x_2) \in L^2 \mid \langle x, x\rangle_L < 0 \text{ and } x_2 > 0\}.$$

The *hyperbolic line* $\mathbb{H}^1$ is the set of all unit future timelike vectors:

$$\mathbb{H}^1 := \{x = (x_1, x_2) \in L^2 \mid \langle x, x\rangle_L = -1 \text{ and } x_2 > 0\}.$$

In other words, $\mathbb{H}^1$ is the upper branch of the hyperbola $x_2^2 = x_1^2 + 1$. It will play in L^2 the same role as the one the unit circle $\mathbb{S}^1$ plays in the Euclidean plane $\mathbb{R}^2$. For any $t \in \mathbb{R}$, let $g(t)$ be the linear isometry of the Lorentzian plane whose matrix in the canonical basis of $\mathbb{R}^2$ is

$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

These isometries $g(t)$ constitute the group G of *hyperbolic translations* of L^2. Note that G is an abelian subgroup of $O(1,1)$ and that $g: \mathbb{R} \to G$, $t \mapsto g(t)$ is a group isomorphism: $g(s+t) = g(s)g(t)$ for all $s, t \in \mathbb{R}$. The hyperbolic line $\mathbb{H}^1$ can be regarded as the orbit of e_2 under the action of G. Any $v(t) = (\sinh t, \cosh t) \in \mathbb{H}^1$ is identified with the unique $t \in \mathbb{R}$ such that $g(t)(e_2) = v(t)$. For any $x, y \in \mathbb{H}^1$, the *oriented hyperbolic angle from x to y* is the unique t such that $g(t)(x) = y$.

A (smooth) curve in L^2 is a differentiable map $c: I \subset \mathbb{R} \to L^2$, where I is an open interval. A curve $c: I \to L^2$ is said to be *regular at t* if $c'(t) \neq 0$. The curve is said to be *regular* if it is regular at any $t \in I$. It is is said to be *spacelike* (resp. *lightlike*, *timelike*) at t if $c'(t)$ is a spacelike (resp. null or lightlike, timelike) vector. It is said to be *spacelike* (resp. *timelike*) if it is spacelike (resp. timelike) at any $t \in I$.

Let $\sigma: L^2 \to L^2$ be the anti-isometric involutive operator given by $\sigma(x_1, x_2) = (x_2, x_1)$. For any $x \in L^2 - \{(0,0)\}$, let $x^\perp := \varepsilon(x)\sigma(x)$. For any nonlightlike $x \in L^2 - \{(0,0)\}$, $(x, x^\perp)$ is a positively oriented (*i.e.,* endowed with the orientation of the canonical basis of $\mathbb{R}^2$) basis of L^2.

Let $c: I \to L^2$ be a spacelike (resp. timelike) curve of class C^2. At any point of $c: I \to L^2$, we can define the *oriented Frenet frame* $(T(t), N(t))$ consisting of *Frenet vectors*

$$T(t) := \frac{c'(t)}{\|c'(t)\|_L} \quad \text{and} \quad N(t) := T(t)^\perp.$$

If $c: I \to L^2$ is parametrized by the pseudo arc length s (that is, if $\|c'(s)\|_L = 1$ for all $s \in I$), then the *algebraic curvature of c* is defined to be the function κ such that $T'(s) = \kappa(s)N(s)$. If it is not the case, a straightforward computation using the fact that $ds/dt = \|c'(t)\|_L$ shows that the algebraic curvature is given by

$$\kappa(t) := \frac{\langle c'(t), \sigma(c''(t))\rangle_L}{\|c'(t)\|_L^3}.$$

If $c: I \to L^2$ is a spacelike hedgehog $x_h: \mathbb{H}^1 \to L^2$ with support function $h \in C^3(\mathbb{R}; \mathbb{R})$, then $c' = x_h' = R_h v'$, where $R_h := h'' - h$ is the so-called *curvature function* of $\mathcal{S}_h$. In this case, we obtain

$$T = \operatorname{sgn}(R_h)v', \quad N = \operatorname{sgn}(R_h)v, \quad \text{and} \quad \kappa(t) = |R_h|^{-1}.$$

3 Evolute

3.1 Evolute of a Spacelike Hedgehog $\mathcal{S}_h$ of L^2

In this subsection, $h \in C^3(\mathbb{R}; \mathbb{R})$. The evolute of $\mathcal{S}_h$ can be defined in two different but equivalent ways: as an envelope or as a locus.

3.1.1 Evolute of $\mathcal{S}_h$ as the envelope of its normal lines

For every $t \in \mathbb{R}$, the support line $L_h(t)$, with coorienting unit normal vector $\nu(t) := (\sinh t, \cosh t)$, has equation $\langle x, \nu(t)\rangle_L := h(t)$. Let $N_h(t)$ be the line through x that is orthogonal (with respect to the Lorentzian metric $\langle\cdot,\cdot\rangle_L$) to $L_h(t)$ in L^2. It will be called the normal line to $\mathcal{H}_h$ at $x_h(t)$. This normal line has equation $\langle x, \nu'(t)\rangle_L := h'(t)$.

Define the *evolute* $\mathcal{D}(\mathcal{S}_h)$ of the spacelike hedgehog $\mathcal{S}_h \subset L^2$ to be the envelope of the family $(N_h(t))_{t\in\mathbb{R}}$ of its normal lines. This evolute $\mathcal{D}(\mathcal{S}_h)$ is thus the curve in L^2 parametrized by $c_h : \mathbb{R} \to L^2, t \longmapsto c_h(t)$, where $c_h(t)$ is the unique solution of the system

$$\begin{cases} \langle x, \nu'(t)\rangle_L := h'(t) \\ \langle x, \nu(t)\rangle_L := h''(t), \end{cases}$$

that is, $c_h(t) = h'(t)\nu'(t) - h''(t)\nu(t)$.

3.1.2 Evolute of $\mathcal{S}_h$ as the locus of its centers of curvature

The evolute of the spacelike hedgehog $\mathcal{S}_h \subset L^2$ can also be defined as the locus of all its centers of curvature. Recall that $x_h'(t) = R_h(t)\nu'(t)$ for all $t \in \mathbb{H}^1 \simeq \mathbb{R}$. Since $\mathcal{S}_h := x_h(\mathbb{H}^1)$ is an envelope parametrized by its coorienting unit normal vector field, the center of curvature of $\mathcal{S}_h$ at $x_h(t)$ is defined to be $c_h(t) := x_h(t) - R_h(t)\nu(t) = h'(t)\nu'(t) - h''(t)\nu(t)$ for all $t \in \mathbb{H}^1$. Of course, if x_h is regular at t, then

$$c_h(t) = x_h(t) - \frac{1}{\kappa(t)} N(t),$$

but the *center of curvature* $c_h(t)$ is well defined even if $x_h'(t) = 0$.

3.2 Timelike Hedgehogs of L^2 and their Evolutes

3.2.1 Definitions

We can also define timelike hedgehogs of L^2. The *timelike hedgehog* with support function $h \in C^1(\mathbb{R};\mathbb{R})$ is defined to be the envelope $\mathcal{T}_h$ of the family $(L_h'(t))_{t\in\mathbb{R}}$ of cooriented timelike lines with equation

$$\langle x, \nu'(t)\rangle_L := h(t), \tag{3.1}$$

$\nu'(t) = (\cosh t, \sinh t)$ being the unit coorienting normal vector of $L_h'(t)$. Partial differentiation of (3.1) yields

$$\langle x, \nu(t)\rangle_L := h'(t). \tag{3.2}$$

From (3.1) and (3.2), the parametrization of $\mathcal{T}_h$ is

$$y_h : \mathbb{R} \to L^2, \quad t \mapsto h(t)\nu'(t) - h'(t)\nu(t).$$

Note that $y_h'(t) = -R_h(t)\nu(t)$ for all $t \in \mathbb{R}$, where $R_h := h'' - h$. The *evolute* $\mathcal{D}(\mathcal{T}_h)$ of a timelike hedgehog $\mathcal{T}_h \subset L^2$ is defined to be the envelope of the family of its normal lines (*i.e.*, of the lines with equation $\langle x, v(t)\rangle_L := h'(t)$) or, equivalently, the locus of its centers of curvature,

$$d_h(t) := y_h(t) - \left(-R_h(t)v'(t)\right) = h''(t)v'(t) - h'(t)v(t), \quad (t \in \mathbb{R}).$$

3.2.2 Relationship between $\mathcal{S}_h = x_h(\mathbb{R})$ and $\mathcal{T}_h = y_h(\mathbb{R})$

Let Σ be the anti-isometric involutive operator of L^2 that is given by $\Sigma(x) = -\sigma(x)$ for all $x \in L^2$. Note that $\Sigma \circ v = -v'$ and $\Sigma \circ v' = -v$.

Proposition 3.1 *For any $h \in C^1(\mathbb{R};\mathbb{R})$, the spacelike hedgehog $\mathcal{S}_h$ and the timelike hedgehog $\mathcal{T}_h$ are related by $\mathcal{T}_h = \Sigma(\mathcal{S}_h)$ and $\mathcal{S}_h = \Sigma(\mathcal{T}_h)$.*

Proof Indeed, their respective parametrizations $x_h := h'v' - hv$ and $y_h := hv' - h'v$ are such that $y_h = \Sigma \circ x_h$ and $x_h = \Sigma \circ y_h$. ■

3.3 Second Evolute

Proposition 3.2 *For any $h \in C^2(\mathbb{R};\mathbb{R})$, the evolute of the spacelike hedgehog $\mathcal{S}_h$ (resp. of the timelike hedgehog $\mathcal{T}_h$) can be given by*

$$\mathcal{D}(\mathcal{S}_h) = \Sigma(\mathcal{S}_{h'}) \quad (\textit{resp. } \mathcal{D}(\mathcal{T}_h) = \Sigma(\mathcal{T}_{h'}))$$

and hence by

$$\mathcal{D}(\mathcal{S}_h) = \mathcal{T}_{h'} \quad (\textit{resp. } \mathcal{D}(\mathcal{T}_h) = \mathcal{S}_{h'})$$

from the previous proposition. See Figure 2 for an illustration.

Proof Indeed, $c_h := h'v' - h''v$ (resp. $d_h := h''v' - h'v$) satisfies $\Sigma \circ c_h = -h'v + h''v' = x_{h'}$ (resp. $\Sigma \circ d_h = -h''v + h'v' = y_{h'}$) and hence $c_h = \Sigma \circ x_{h'}$ (resp. $d_h = \Sigma \circ y_{h'}$). ■

Corollary 3.3 *For any $h \in C^3(\mathbb{R};\mathbb{R})$, the second evolute of the spacelike hedgehog $\mathcal{S}_h$ (resp. of the timelike hedgehog $\mathcal{T}_h$) is simply the spacelike (resp. timelike hedgehog) with support function h'':*

$$\mathcal{D}^2(\mathcal{S}_h) := \mathcal{D}(\mathcal{D}(\mathcal{S}_h)) = \mathcal{S}_{h''} \quad (\textit{resp. } \mathcal{D}^2(\mathcal{T}_h) = \mathcal{D}(\mathcal{D}(\mathcal{T}_h)) = \mathcal{T}_{h''}).$$

4 Duality

Let $c\colon I \subset \mathbb{R} \to L^2$ be a spacelike or timelike curve in L^2 and let $p_c\colon I \to L^2$ be its pedal curve. For any $t \in I$, $p_c(t)$ is the foot of the perpendicular from the origin to the tangent line to c at $c(t)$. Note that, replacing tangent lines by support lines, we can define the pedal curve of a spacelike (resp. timelike) hedgehog even if x_h (resp. y_h) is not regular. Assume that $\|c(t)\|_L.\|p_c(t)\|_L \neq 0$ for all $t \in I$. Define the *star curve of c* to be the curve $c^*\colon I \subset \mathbb{R} \to L^2$ given by $c^* := i \circ p_C$, where

$$i(x) := \varepsilon(x)\frac{x}{\|x\|_L^2} \quad \text{for all } x \in L^2 \text{ such that } \|x\|_L \neq 0$$

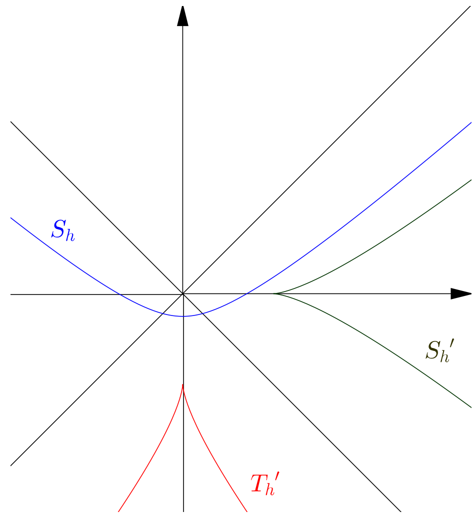

Figure 2: $\mathcal{S}_h$ and $\mathcal{T}_{h'}$ if $h(t) := \cosh(2t)$.

(recall that $\varepsilon(x) := \operatorname{sgn}(\langle x, x\rangle_L)$). If $c\colon I \to L^2$ is the restriction of a spacelike hedgehog x_h (resp. a timelike hedgehog y_h) to I, then $p_c = -hv$ (resp. $p_c = hv'$), and, assuming that $h.\|x_h\|_L \neq 0$ (resp. $h.\|y_h\|_L \neq 0$), we can define its star curve in the same way.

Proposition 4.1 *Let I be an open interval of $\mathbb{R}$. If $c\colon I \to L^2$ is the restriction to I of a spacelike hedgehog x_h (resp. a timelike hedgehog y_h) such that $h.\|x_h\|_L \neq 0$ (resp. $h.\|y_h\|_L \neq 0$) on I, then $(c^*)^* = c$.*

Proof If $c = x_h$ (resp. $c = y_h$), then $p_c = -hv$ (resp. $p_c = hv'$). Thus,

$$x_h^* = \frac{v}{h} \quad \left(\text{resp. } y_h^* = \frac{v'}{h}\right).$$

Differentiation gives

$$(x_h^*)' = \frac{y_h}{h^2} \quad \left(\text{resp. } (y_h^*)' = -\frac{x_h}{h^2}\right).$$

Now

$$x_h^* = \frac{h' y_h - h x_h}{h(h^2 - (h')^2)} \quad \left(\text{resp. } y_h^* = \frac{h y_h - h' x_h}{h(h^2 - (h')^2)}\right).$$

Therefore

$$p_{x_h^*} = \frac{x_h}{(h')^2 - h^2} \quad \left(\text{resp. } p_{y_h^*} = \frac{y_h}{h^2 - (h')^2}\right),$$

and hence $(x_h^*)^* = x_h \quad (\text{resp. } (y_h^*)^* = y_h)$. ∎

Definition 4.2 For any $h \in C^1(\mathbb{R};\mathbb{R})$ such that $h.\|x_h\|_L \neq 0$ (resp. $h.\|y_h\|_L \neq 0$), we shall say that $\mathcal{S}_h^* := x_h^*(\mathbb{R})$ (resp. $\mathcal{T}_h^* := y_h^*(\mathbb{R})$) is the dual curve of the spacelike (resp. timelike) hedgehog $\mathcal{S}_h$ (resp. $\mathcal{T}_h$).

5 Convolution

Differences of convex bodies of $\mathbb{R}^2$ do not only constitute a real vector space $(\mathcal{H}^2, +, \cdot)$ but also a commutative and associative $\mathbb{R}$-algebra. As noticed by Görtler [4, 5], we can define the convolution product of two hedgehogs $\mathcal{H}_f$ and $\mathcal{H}_g$ of $\mathbb{R}^2$ as the hedgehog with support function

$$(f * g)(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta - \alpha)g(\alpha)\,d\alpha,$$

for all $\theta \in \mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ and we can check at once that $(\mathcal{H}^2, +, \cdot, *)$ is then a commutative and associative algebra. The point of interest is, of course, that the convolution product inherits many properties of the factors [10, Section 6]. In particular, H. Görtler noticed that the convolution product of two plane convex bodies is still a plane convex body. The purpose of this section is to give a similar result for Fuchsian hedgehogs.

Let $h \in C^2(\mathbb{H}^1;\mathbb{R})$. Recall that $x_h'(t) = R_h(t)v'(t)$ for all $t \in \mathbb{H}^1$ Hence the spacelike hedgehog $\mathcal{S}_h = x_h(\mathbb{H}^1)$ is a regular curve if and only if its curvature function R_h is everywhere nonzero. In that case, $\mathcal{S}_h$ is said to be *convex*.

Definition 5.1 Let $h \in C^2(\mathbb{H}^1;\mathbb{R})$. The spacelike hedgehog $\mathcal{S}_h$ is said to be *convex* if its curvature function $R_h := h'' - h$ is everywhere nonzero on $\mathbb{H}^1$. It is said to be *future convex* (resp. *past convex*) if its curvature function is everywhere positive (resp. negative) on $\mathbb{H}^1$.

Definition 5.2 Let $h \in C^2(\mathbb{H}^1/\Gamma;\mathbb{R})$. The Γ-hedgehog Γ_h is said to be a Γ-hedgehog of class C_+^2 of $F = \{x = (x_1, x_2) \in L^2 \mid \langle x, x\rangle_L < 0 \text{ and } x_2 > 0\}$ if $h < 0$ and $R_h > 0$.

Remark 5.3 A Γ-hedgehog of class C_+^2 of F can indifferently be regarded as a convex curve on F or as a convex closed curve on F/Γ.

Definition 5.4 Let Γ_f and Γ_g be Γ-hedgehogs whose respective support functions f and g are in $C^1(\mathbb{H}^1/\Gamma;\mathbb{R})$. The convolution of Γ_f and Γ_g is the Γ-hedgehog Γ_{f*g} with support function

$$(f * g)(t) = -\int_0^T f(t - s)g(s)\,ds \quad \textit{for all } t \in [0, T].$$

The operation of convolution of Γ-hedgehogs is of course commutative, associative, and distributive over addition. Here is an analogous result of Görtler's theorem.

Proposition 5.5 *Let Γ_f and Γ_g be Γ-hedgehogs whose respective support functions f and g are in $C^2(\mathbb{H}^1/\Gamma;\mathbb{R})$. If Γ_g is a Γ-hedgehog of class C_+^2 of F and if f is negative, then Γ_{f*g} is a Γ-hedgehog of class C_+^2 of F.*

Proof If $f < 0$, $g < 0$ and $R_g > 0$, then $(f * g) < 0$ and $R_{f*g} > 0$. Indeed, the first inequality is trivial and an easy computation shows that

$$\begin{aligned} R_{f*g}(t) &= (f * g)''(t) - (f * g)(t) = (f * g'')(t) - (f * g)(t) \\ &= (f * (g'' - g))(t) = (f * R_g)(t) = -\int_0^T f(t-s) R_g(s)\, ds \end{aligned}$$

is positive for all $t \in [0, T]$, since $f < 0$ and $R_g > 0$. ∎

Corollary 5.6 *If Γ_f and Γ_g are Γ-hedgehogs of class C^2_+ of F, then Γ_{f*g} is also a Γ-hedgehog of class C^2_+ of F.*

6 Isometric Excess and Area of the Evolute

The following theorem is analogous to Theorem 1.1 for Fuchsian hedgehogs.

Theorem 6.1 *Let $T \in]0, 2\pi]$. For any T-periodic $h \in C^3(\mathbb{R};\mathbb{R})$,*

$$0 \le 2Ta(h) - l(h)^2 \le 2Ta(h'),$$

where $l(h)$ and $a(h)$ are respectively the signed length and area of Γ_h and $a(h')$ the signed area of its evolute.

Proof The first inequality is simply the isoperimetric inequality (1.8). Let us prove the second one. First note that

$$a(h) - a(h') = \frac{1}{2}\Big(\int_0^T h^2 dt - \int_0^T (h'')^2 dt \Big).$$

Let $a_n(f)$ and $b_n(f)$ denote the Fourier coefficients of $f \in C^1(\mathbb{R};\mathbb{R})$:

$$\begin{aligned} a_0(f) &:= \frac{1}{T}\int_0^T f(t)dt, \\ a_n(f) &:= \frac{2}{T}\int_0^T f(t)\cos n\omega t\, dt, \\ b_n(f) &:= \frac{2}{T}\int_0^T f(t)\sin n\omega t\, dt, \end{aligned}$$

where $\omega := 2\pi/T$ and $n \in \mathbb{N}^*$. Recall that $a_n(h'') = -(n\omega)^2 a_n(h)$ and $b_n(h'') = -(n\omega)^2 b_n(h)$ for all $n \in \mathbb{N}^*$. Parseval's equality gives

$$\frac{1}{2}\int_0^T (h^2 - (h'')^2)(t)dt = \frac{T}{2}a_0(h)^2 + \frac{T}{4}\sum_{n=1}^{+\infty}(1 - (n\omega)^4)(a_n(h)^2 + b_n(h)^2).$$

Now the sum in the right-hand side is obviously nonpositive, so that

$$\frac{1}{2}\int_0^T (h^2 - (h'')^2)(t)dt \le \frac{T}{2}a_0(h)^2 = \frac{l(h)^2}{2T}.$$

Hence $2Ta(h) - l(h)^2 \le 2Ta(h')$, which achieves the proof. ∎

Remark 6.2

(a) For $T \in \,]0, 2\pi[$, the equality $2Ta(h) - l(h)^2 = 2Ta(h')$ holds if and only if h is constant.

(b) For $T = 2\pi$, the equality $2Ta(h) - l(h)^2 = 2Ta(h')$ may hold for nonconstant $h \in C^3(\mathbb{H}^1/\Gamma; \mathbb{R})$. Consider, for instance, $h(t) := \cos t$.

(c) The assumption $T \in]0, 2\pi]$ is necessary even if we restrict to Γ-hedgehogs that are Γ-hedgehogs of class C^2_+ of F. Consider, for instance, $h(t) := -2 + \cos(\frac{2\pi}{7} t)$, which is such that $h < 0$ and $R_h > 0$.

7 Reversed Bonnesen Inequality

Let K be a convex body with non-empty interior in $\mathbb{R}^2$. In the 1920's, T. Bonnesen gave various sharpenings of the isoperimetric inequality

$$A \leq \frac{L^2}{4\pi},$$

where L and A denote respectively the perimeter and the area of K. In particular, he proved the inequality

$$L^2 - 4\pi A \geq \pi^2 (R - r)^2, \tag{7.1}$$

where r and R are respectively the inradius and the circumradius of K (*i.e.,* the radii of the largest inscribed and the smallest circumscribed circles of the boundary of K, respectively). He further proved that the equality holds in (7.1) if and only if $R = r$, *i.e.,* if K is a disc. The proof by Bonnesen is reproduced in [1, pp. 108-110]. For a survey of Bonnesen-type inequalities in Euclidean spaces, we refer the reader to [12]. Let us prove the following reversed Bonnesen inequality for Fuchsian hedgehogs.

Theorem 7.1 *For any T-periodic $h \in C^2(\mathbb{R}; \mathbb{R})$,*

$$\frac{1}{2T}(R - r)^2 \leq a(h) - \frac{l(h)^2}{2T},$$

where $l(h)$ and $a(h)$ are respectively the signed length and area of Γ_h and where $r := \min_{0 \leq t \leq T}(-h(t))$ and $R := \max_{0 \leq t \leq T}(-h(t))$. Furthermore, the equality holds if and only if $R = r$.

Proof Since $-h\colon \mathbb{R} \to \mathbb{R}$ is continuous, there exists $(t_0, t_1) \in [0, T]^2$ such that $r = -h(t_0)$ and $R = -h(t_1)$. Thus we have

$$(R - r)^2 = (h(t_1) - h(t_0))^2 = \left(\int_{t_0}^{t_1} h'(t) dt \right)^2.$$

By the Cauchy–Schwarz inequality, we deduce that

$$(R - r)^2 \leq |t_1 - t_0| \int_{\min(t_0, t_1)}^{\max(t_0, t_1)} h'(t)^2 dt \leq T \int_0^T h'(t)^2 dt.$$

Now

$$\int_0^T h'(t)^2 dt = 2a(h) - \int_0^T h(t)^2 dt$$

and again by the Cauchy-Schwarz inequality

$$l(h)^2 = \left(\int_0^T h(t)dt \right)^2 \leq T \int_0^T h(t)^2 dt.$$

Therefore $(R-r)^2 \leq 2Ta(h) - l(h)^2$, which achieves the proof of the reversed Bonnesen inequality. Finally, considering equality cases at each step, we immediately see that the equality holds if and only if h is constant, which completes the proof. ■

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